This examination contains seven problems; you are to complete six of the seven problems. The problem that you choose to omit must be indicated clearly on the front cover sheet. The format for this examination is closed-book, closed-notes and the use of a calculator is not permitted. The time allotted for this examination is three hours.

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NAME: ___________________________________________
Problem One (20 points) – Computer Science Concepts

Name: _____________________________

Part A:

1. (10 points) Prepare a class diagram from the instance diagram in Figure 1.

![Figure 1 Instance Diagram](image-url)
2. (10 points) Prepare a data flow diagram for computing the volume and surface area of a cylinder. Inputs are the height and radius of the cylinder. Outputs are volume and surface area.
Problem Two (20 points) – Mathematical Concepts

Name: ______________________________

Part A
1. Ordinary Differential Equations (10 points)

1) An important modeling construct, especially for many physics-based systems, is the ordinary differential equation. You are requested to solve the following ordinary differential equation.

$$\frac{dx(t)}{dt} + 2x(t) = 2t + 3; \quad x(0) = 2.$$  

2) Define appropriate state variables and then express the following second-order differential equation as two first-order differential equations; that is, express the problem in state variable form.

$$\frac{d^2x(t)}{dt^2} + 3\frac{dx(t)}{dt} + 2x(t) = 4; \quad x(0) = 2, \quad \frac{dx(t)}{dt} \bigg|_{t=0} = 3.$$  

2. Linear Algebra (10 points)

Important concepts from linear algebra are the notions of vector space and linear transformation.

1) Consider the vectors \( \{f_1, f_2, f_3\} \in \mathbb{R}^3 \) where \( f_1 = (1,1)^T \), \( f_2 = (1,-1,0)^T \), and \( f_3 = (1,1,-2)^T \). Show that these vectors are a basis for \( \mathbb{R}^3 \).

2) Formally define a “linear transformation \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \).”

3) Suppose \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) is a transformation defined by

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y + z \\ x + z \\ x + y \end{bmatrix}.$$  

Demonstrate that \( T \) is a linear transformation.

4) An important property of linear transformations is presented in the following theorem.

**Theorem.** Let \( L : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Then there exists a unique \( m \times n \) matrix \( A \) such that \( L(u) = Au \) for \( u \in \mathbb{R}^n \).

Determine the matrix representation \( A \) for the linear transformation \( T \) (from 3 above) relative the basis set \( \{f_1, f_2, f_3\} \in \mathbb{R}^3 \) (from 1 above).
Problem Three (20 points) – Probability and Statistics

Name: ______________________________

Part A
1. Statistical Distributions (10 Points)

1) Consider the Negative Binomial Distribution:
   a) What does it traditionally represent?
   b) Is it a continuous or discrete distribution?

2) In terms of input analysis, when might you use the triangular distribution?

3) Show that if \( X \sim N(0,1) \) and \( Z = \sigma \cdot X + \mu \) then \( Z \sim N(\mu, \sigma^2) \) in terms of cumulative distribution function.

The probability density function for a normal distribution, \( N(\mu, \sigma^2) \), is given by:

\[
f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2}(x - \mu)^2 \right\}
\]

2. Probability (10 Points)

The probability of some event ‘A’ occurring is represented by \( P(A) \). The conditional probability of event ‘A’ occurring given some event ‘B’ has already occurred is represented by \( P(A \mid B) \). The joint probability of events ‘A’ and ‘B’ occurring is represented by \( P(A, B) \). Bayes' theorem states that:

\[ P(A \mid B).P(B) = P(A, B) \]

The expected value for a random variable ‘X’ is represented by \( E(X) \) and the conditional expected value of ‘X’, given some event ‘B’, is represented by \( E(X \mid B) \). ‘X’ and ‘Y’ are finite discrete random variables.

4) Write out the expected formula for ‘X’: \( E(X) \)

5) Write out the conditional expected formula: \( E(X \mid Y = y) \)

6) Prove the tower rule for ‘X’ and ‘Y’: \( E(X) = E(E(X \mid Y)) \). Marks will be deducted if any steps are missed out.
Problem Four (20 points) – Discrete Event Simulation

Name: _______________________________

Part A

1. Statistical Models: 10 points
A recent survey indicated that 82% of single women aged 25 years old will be married in their lifetime. Using the binomial distribution, find the probability that two or three women in a sample of twenty will never be married.

Discrete distribution formulas

1) Bernoulli
Consider an experiment consisting of \( n \) trials, each of which can be a success or failure. Let \( X_j=1 \) if the \( j \)th experiment results in a success, and \( X_j=0 \) if the \( j \)th experiment results in a failure. Assuming that the trials are independent and that the probability of success is constant from trial to trial, the probability of success of the \( j \)th trial is

\[
p_j(x_j) = p(x_j) = \begin{cases} \frac{p}{1-p} = q & x_j = 1, j = 1,2, \ldots, n \\ 0 & x_j = 0, j = 1,2, \ldots, n \\ \end{cases}
\]

The mean and variance are calculated as follows:

\[
E(X_j) = 0 \cdot q + 1 \cdot p = p \quad \text{and} \quad V(X_j) = [0^2 \cdot q + 1^2 \cdot p] - p^2 = p(1-p).
\]

2) Binomial
The random variable \( X \) that denotes the number of successes in \( n \) Bernoulli trials has a binomial distribution given by \( p(x) \), where

\[
p(x) = \binom{n}{x} p^x q^{n-x} \quad x = 1, j = 1,2, \ldots, n
\]

3) Geometric & Negative Binomial
The geometric distribution is related to a sequence of Bernoulli trials; the random variable of interest, \( X \), is defined to be the number of trials to achieve the first success. The distribution of \( X \) is given by

\[
p(x) = \begin{cases} q^{x-1}p & x = 1,2, \ldots \\ 0 & \text{otherwise} \end{cases}
\]

More generally, the negative binomial distribution of the distribution of the number of trials until the \( k \)th success, for \( k=1,2,\ldots \). If \( Y \) has a negative binomial distribution with parameters \( p \) and \( k \), then the distribution of \( Y \) is given by:
4) Poisson

The Poisson probability mass function is given by

\[ p(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & x = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \]

2. Queuing Models: 10 points

Arrivals to a self-service gasoline pump occur in a Poisson fashion at a rate of 12 per hour. Service time has a distribution which averages 4 minutes with a standard deviation of 1.33 minutes. What is the expected number of vehicles in the system? No need to work out the solution: simply show the work prior to the solution, as would be input into a calculator.

Queue Model Formulas:

**M/G/1 queues**

\[ \rho = \frac{\lambda}{\mu}, \quad P_0 = 1 - \rho \]
\[ L = \rho + \frac{\rho^2(1 + \sigma^2 \mu^2)}{2(1 - \rho)}, \quad L_Q = \frac{\rho^2(1 + \sigma^2 \mu^2)}{2(1 - \rho)} \]
\[ w = \frac{1 + \frac{\lambda}{\mu}(1 + \sigma^2 \mu^2)}{2(1 - \rho)}, \quad w_Q = \frac{\lambda(1 + \mu^2 + \sigma^2)}{2(1 - \rho)} \]

**M/M/c queues**

\[ \rho = \frac{\lambda}{c \mu} \]
\[ P_0 = \left[ \sum_{n=0}^{c-1} \frac{(\lambda/\mu)^n}{n!} \right] \left[ \frac{\lambda^c}{c!} \left( \frac{c \mu}{c \mu - \lambda} \right) \right]^{-1} \]
\[ L = c \rho + \frac{(c \rho)^{c+1} P_0}{c(c!)(1 - \rho)^2} = c \rho + P \frac{P_0(L(\infty) \geq c)}{1 - \rho} \]
\[ w = \frac{L}{\lambda}, \quad w_Q = w - \frac{1}{\mu} \]

**M/M/1 queues**

\[ \rho = \frac{\lambda}{\mu}, \quad P_n = (1 - \rho) \rho^n \]
\[ L = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1 - \rho}, \quad L_Q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{\mu(1 - \rho)} \]
\[ w = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1 - \rho)}, \quad w_Q = \frac{\lambda}{\mu(\mu - \lambda)} \]
Problem Five (20 points) – System Modeling Concepts

Name: _______________________________

Part A
Fill in the blanks by choosing the best answer from the following: Aggregation, Association, Class, Class Diagram, Collaboration Diagram, Computer Simulation, Contention, Dependency, FEA, Formalism, FSA, Inheritance, Interaction Diagram, Markov Model, Model, Multimodel, Multiplicity, Prioritization, Sequence Diagram, Simulation, System, UML, Uniform, Use Case. (Total: 20 points, 1 point/problem).

Note that some words may be used more than once and some may not be used at all.

a) _______________The mechanism by which more specific elements incorporate structure and behavior of more general elements related by behavior.

b) _______________The semantic relationship between two or more classes that specify connections among their instances.

c) _______________Something that is real or potentially real and is concerned with space-time effects and causal relationships among parts.

d) _______________A special form of association that specifies a whole-part relationship between the aggregate and the component part.

e) _______________An abstraction from reality that describes a dynamic system.

f) _______________An applied methodology in which the behavior of complex systems is described using mathematical or symbolic symbols.

g) _______________The discipline of designing a model of a system, executing the model on a digital computer, and analyzing the execution output.

h) _______________A distribution used in generating random numbers.

i) _______________A generic term that applies to several types of diagrams that emphasize object interactivity.

j) _______________Something we use in lieu of the real thing in order to understand something about that real thing.

k) _______________A collection of individual models – each characterizing an abstraction level – connected together in a seamless fashion to promote level traversal.

l) _______________A diagram that shows interactions organized around the structure of the model using either classifiers and associations or instances and links.

m) _______________A description of a set of objects that share the same attributes, operations, and semantics.

n) _______________An unambiguous description of the semantics of a model.

o) _______________A graphical modeling language used in aiding the unambiguous description of a model.
p) A diagram that shows a collection of declarative and static model elements such as classes, datatypes, their contents and their relationships.

q) A relationship between two modeling elements in which a change to one will affect change in the other.

r) A specification of the range of allowable cardinalities that a set may assume.

s) A specification of a sequence of actions, including variants, that a system or entity can perform while interacting with actors of the system.

t) A diagram that shows object interactions arranged chronologically. It shows the objects participating in the interaction and the messages exchanged.
Problem Six (20 points) – Analysis Concepts

Name: ______________________________

Part A
1. (5 points) Describe in as much detail as possible how you would generate a random variate from the following probability mass function: \( p(10)=0.2, \ p(25)=0.3, \ p(26)=0.2, \ p(30)=0.3 \). That is, the value 10 has a probability of 0.2 etc.

2. (10 points) We all know that the confidence interval for the population mean is given by the formula:

\[
\bar{X}_n \pm t_{n-1,\alpha} \frac{S_n}{\sqrt{n}}
\]

Derive this result from scratch. Clearly state your assumptions.

3. (5 points) Describe in words how one should interpret the above confidence interval.
Problem Seven (20 points) – Visualization Concepts

Name: _______________________________

Part A:

1. (10 points)
   1) What is the Z buffer (or depth buffer) in OpenGL? Explain how it is used. (2 pts)
   2) What is the difference between perspective projection and orthographic projection? What commands do you use in OpenGL to set perspective projection and orthographic projection? (2 pts)
   3) Explain why we can use combinations of red, green, and blue to represent most visible colors. (2 pts)
   4) Derive the matrix that converts world coordinates to camera coordinates determined by the OpenGL command gluLookAt(0, 0, 5, 0, 0, 0, 0, 1, 0). Show both your work and final results. (4 pts)

2. (10 points)
   1) Give an example of a matrix that represents a scaling operation and calculate its inverse. Explain the main uses of scaling in visualization. (2 pts)
   2) Explain why triangles are the most commonly used polygons for 3D meshes in computer graphics and visualization. (2 pts)
   3) Calculate the dot product of two vectors $\mathbf{a} = (1,2,3)^T$ and $\mathbf{b} = (2,2,-2)^T$. What can you say about the relationship between these two vectors based on their dot product? (2 pts)
   4) Compute the final transformation matrix for the following transformations. Show both your work and final results. (4 pts)
      A. An object is first translated by a displacement vector (2, 2, 2) and then scaled by the scaling factor (3, 3, 3);
      B. An object is first scaled by a scaling factor (3, 3, 3) and then translated by a displacement vector (2, 2, 2).